Controlling chaos in nonlinear three-wave coupling

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It is shown that the chaotic orbits in the saturated states of nonlinear three-wave coupling can be controlled by applying a new resonance, via a small sinusoidal anti-Stokes wave, to the system. For a given set of linear frequency mismatch and growth-damping parameters, any desired periodic orbit can be achieved by a proper adjustment of the amplitude of the control wave. This method is illustrated by numerical simulations. [S1063-651X(96)08106-8]

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Nonlinear three-wave coupling is of general interest in many branches of physics such as nuclear fusion, space geophysics, astrophysics, nonlinear optics, and fluid mechanics. For example, it causes the stimulated scattering and anomalous absorption of laser beams in inertial fusion experiments [1] and appears in the plasma edge region of a magnetic fusion device during radio-frequency heating experiments [2]; it is responsible for the generation and modulation of plasma waves in the planetary magnetosphere and solar wind [3]; it might be the origin of pulsar eclipse [4]; it provides tunable lasers based on optical parametric oscillators [5]; it arises on shear flows and in interactions between interfacial and gravity waves in hydrodynamics [6].

Chaos is very common in nature and in man-made devices. Recently, there has been a growing interest in developing methods to control chaos in nonlinear dynamical systems. In this paper, we present a method of controlling chaos in nonlinear three-wave coupling.

One of the first approaches introduced to eliminate chaos and achieve a desired behavior in a dynamical system involved large perturbations which completely changed its dynamics [7,8]. Later, it was suggested that the chaotic behavior of a system may be controlled by applying small time-dependent perturbations without creating new orbits that have very different properties from the existing ones [9]. This idea is based on the fact that a chaotic attractor typically has embedded in it an infinite number of unstable periodic orbits [10]. This inherent property of chaotic attractor gives us the flexibility to choose the most desirable periodic orbit among a large number of periodic orbits in the attractor, by introducing a small perturbation to an arbitrary accessible parameter of the system.

The method of applying small perturbations in chaotic systems to obtain a desired regular asymptotic state has been applied to a variety of physical applications. In an experiment consisted of a gravitational buckled amorphous magnetoelastic ribbon, chaotic behavior was controlled by using an external magnetic field [11,12]. The spin-wave instability pumped by microwave radiation can be suppressed by small time-dependent modulations in the biasing magnetic field [13]. Control of chaos displayed by the thermal convection in which the heating rates exceed a certain threshold was performed via small adjustments of the heating rate [14]; the theoretical modeling of this experiment based on the well-known Lorenz equations [15] is in good agreement with the experimental results. The targeting of orbits to a desired state was shown to be particularly simple when the system can be described by a one-dimensional map or even when one wants to reach a stationary state in a three-dimensional flow [16,17]. The possibility of using chaos to transmit information encoding a message in the controlled symbolic dynamics of a chaotic oscillator was demonstrated recently [18].

The simplest model for describing the temporal dynamics of resonant nonlinear coupling of three waves can be obtained assuming that the nonlinearity is sufficiently weak so that only quadratic terms in the wave amplitudes need to be considered. Moreover, the waves may be assumed monochromatic, with the electric fields written in the form $E_\alpha(\mathbf{x},t)=(1/2)A_\alpha(\mathbf{x},t)\exp i(\mathbf{k}_\alpha\cdot\mathbf{x}-\omega_\alpha t)+\text{c.c.}$ (where $\alpha=1,2,3$) and the time scale of the nonlinear interactions is much longer than the periods of the linear (uncoupled) waves [i.e., $\omega^{-1}\ll A(\partial_t A)^{-1}$]. In order for three-wave interactions to occur, the wave frequencies ω_α and wave vectors $\mathbf{k}_\alpha$ must satisfy the resonant conditions

$$\omega_3\approx\omega_1-\omega_2,\quad \mathbf{k}_3=\mathbf{k}_1-\mathbf{k}_2. \tag{1}$$

Under these circumstances, the nonlinear temporal dynamics of the system can be governed by the following set of three first-order autonomous differential equations written in terms of the complex slowly varying wave amplitude [19–21]:

$$\dot{A}_1=\nu_1' A_1+A_2A_3, \tag{2}$$

$$\dot{A}_2=i\,\delta A_2+\nu_2' A_2-A_1A_3^*, \tag{3}$$

$$\dot{A}_3=\nu_3' A_3-A_1A_2^*, \tag{4}$$

where the dot denotes differentiation with respect to the timelike variable $\tau=\chi t$, χ is a characteristic frequency; $\delta=(\omega_1-\omega_2-\omega_3)/\chi$ is the normalized linear frequency mismatch and $\nu_\alpha'=\nu_\alpha/\chi$ give the linear wave behaviors on the long time scale (i.e., either growth or damping, depending on the sign of ν_α). We assume here that the wave A_1 is linearly unstable ($\nu_1>0$) and the other two waves, A_2 and A_3, are linearly damped ($\nu_2'=\nu_3'\equiv-\nu<0$) and henceforth we set $\chi=\nu_1$ so that $\nu_1'=1$ [19–21].

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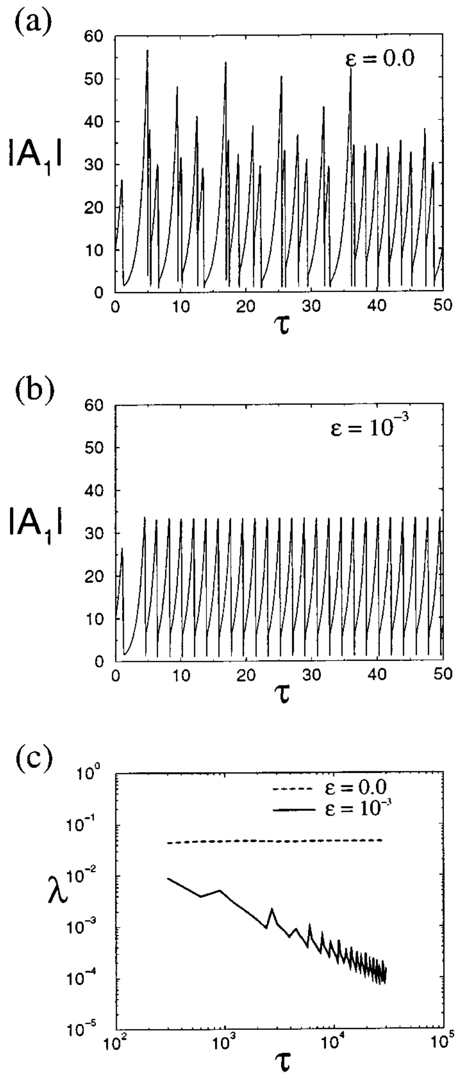

FIG. 1. The plot of (a) the chaotic time series $|A_1(\tau)|$ for $\epsilon=0$, (b) the controlled periodic time series $|A_1(\tau)|$ for $\epsilon=10^{-3}$, (c) the behavior of the maximum Lyapunov exponent λ for $\epsilon=0$ and 10^{-3}. The linear frequency mismatch and growth-damping parameters are $\delta=2$ and $\nu=15$.

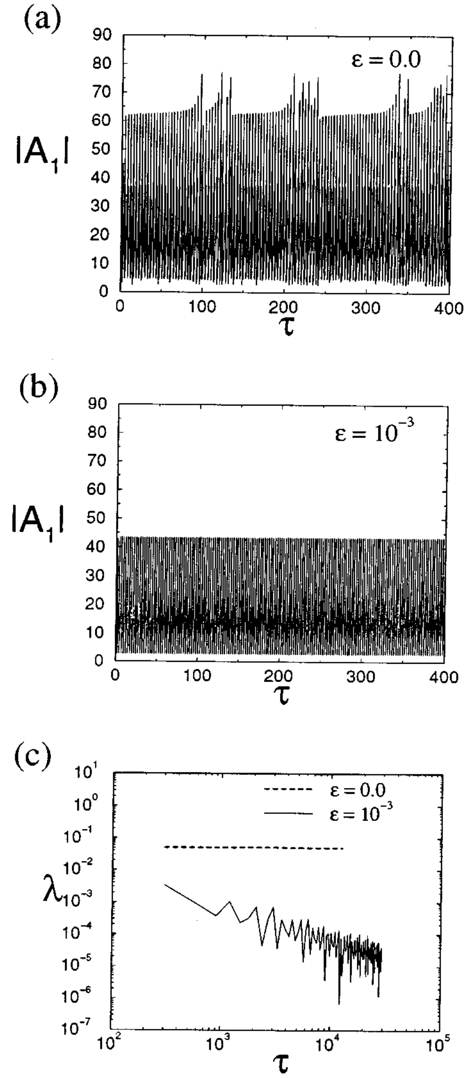

FIG. 2. The plot of (a) the chaotic time series $|A_1(\tau)|$ for $\epsilon=0$, (b) the controlled periodic time series $|A_1(\tau)|$ for $\epsilon=10^{-3}$, (c) the behavior of the maximum Lyapunov exponent λ for $\epsilon=0$ and $\epsilon=10^{-3}$. The linear frequency mismatch and growth-damping parameters are $\delta=5$ and $\nu=21.61$.

By fixing the parameter δ and varying ν, the system (2)–(4) exhibits a great variety of asymptotic behaviors: divergence, fixed point, limit cycle, and strange attractor [19–21]. The transitions from limit cycles to strange attractors in the system (2)–(4) follow two different routes depending on the value of δ and ν: (i) the limit cycles undergo a cascade of period-doubling bifurcations [19,20]; (ii) the stable periodic cycles abruptly take over the place of the chaotic orbits via the intermittency route [21,22]. Figure 1(a) shows an example of a chaotic time series of $|A_1(\tau)|$ evolved via period-doubling bifurcations [19,20]. Figure 2(a) shows an example of a chaotic time series of $|A_1(\tau)|$ evolved via intermittency [21].

We discuss next a method of controlling the chaotic solutions of Eqs. (2)–(4). For resonant four-wave coupling processes involving two wave triplets, it is known that the presence of the second triplet having two waves in common with the first can increase or stabilize the instability of the first triplet [23]. This idea was extended to resonant four-wave interactions involving negative-energy modes; it was shown that the weaker triplet can be stabilized by the stronger triplet against the explosive instability [24]. We apply the above concepts to control chaos in three-wave coupling. Let us denote the pump wave by the subscript 1, the idler wave by the subscript 2, and the Stokes wave by the subscript 3. This triplet satisfies the resonant condition (1) and evolves according to Eqs. (2)–(4). We now introduce a new resonance into the system through the addition of an anti-Stokes wave, $E_4(\mathbf{x},t)=(1/2)A_4(\mathbf{x},t)\exp i(\mathbf{k}_4\cdot\mathbf{x}-\omega_4 t)+\text{c.c.}$, which in effect adds a second triplet into the system with $(\omega_4,\mathbf{k}_4)$ obeying the following resonant conditions

$$\omega_4=\omega_1+\omega_2, \quad \mathbf{k}_4=\mathbf{k}_1+\mathbf{k}_2. \tag{5}$$

Hence, the system now consists of two wave triplets satisfying the two sets of resonant conditions given by Eqs. (1) and

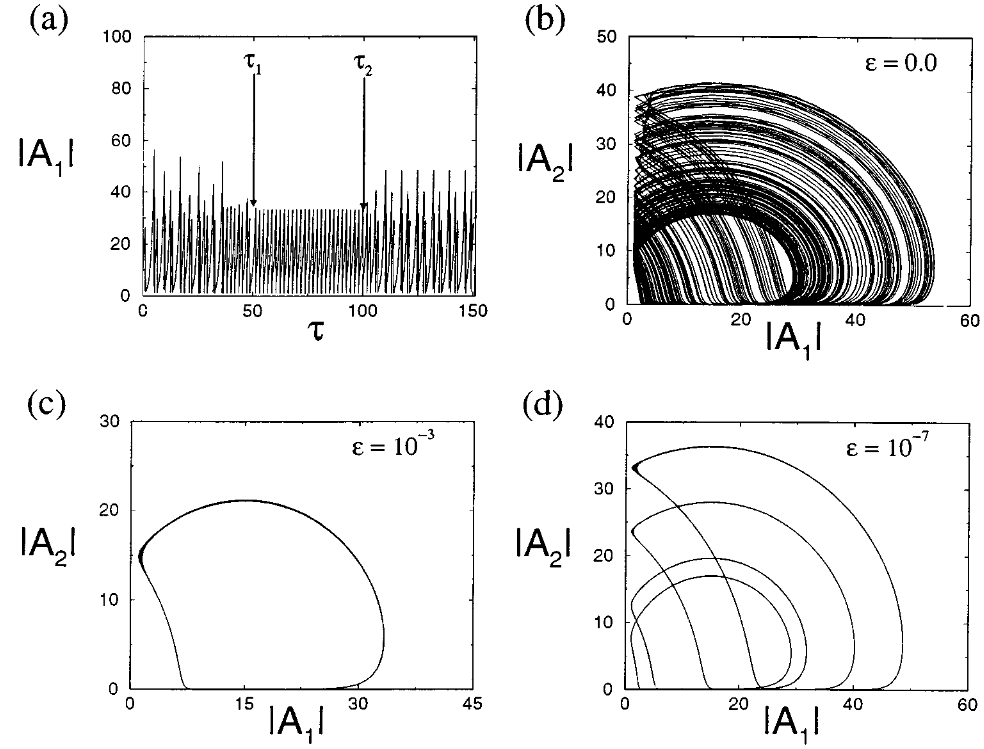

FIG. 3. The plot of (a) the chaotic time series $|A_1(\tau)|$ with $\epsilon=0$ for $0<\tau<\tau_1$, $\epsilon=10^{-3}$ for $\tau_1<\tau<\tau_2$, and $\epsilon=10^{-7}$ for $\tau_2<\tau$; and the corresponding phase portraits for $\epsilon=0$ (b), $\epsilon=10^{-3}$ (c) and $\epsilon=10^{-7}$ (d). The linear frequency mismatch and growth-damping parameters are $\delta=2$ and $\nu=15$.

(5), respectively. The amplitude of this *control* wave is kept *small* so that $|A_4|\equiv|\epsilon|\ll|A_{1,2,3}|$, and *constant* so that $d\epsilon/d\tau=0$. In the presence of the *control* wave, Eqs. (2)–(4) are modified to [23,24]

$$\dot{A}_1=A_1+A_2A_3-\epsilon A_2^*, \tag{6}$$

$$\dot{A}_2=i\delta A_2-\nu A_2-A_1A_3^*+\epsilon A_1^*, \tag{7}$$

$$\dot{A}_3=-\nu A_3-A_1A_2^*. \tag{8}$$

For a given region of the parameter space (δ,ν) where the solution of Eqs. (6)–(8), in the absence of the control wave, is chaotic we optimize the influence of the new resonance (5) in the system by adjusting the *control* parameter ϵ in order to target the system to a desired periodic orbit. This method of control works in both chaotic regions evolved from either period-doubling bifurcation or intermittency. Figures 1(b) and 2(b) show the period-1 orbit, controlled with $\epsilon=10^{-3}$. The characterization of order and chaos is performed by calculating the maximum Lyapunov exponent of the time series [25], as shown in Figs. 1(c) and 2(c). Figures 1(c) and 2(c) show that in the chaotic state $(\epsilon=0)$ λ tends to a positive value as $\tau\rightarrow\infty$; whereas in the controlled regular state $(\epsilon=10^{-3})$ λ tends to zero as $\tau\rightarrow\infty$.

In addition to the ability to bring a given chaotic orbit to a desired periodic orbit, this method of controlling chaos also enables us to choose the time interval during which a desired ordered state is preferred. Figures 3(a) and 4(a) illustrate the above two features in the same graph, for transition to chaos via period-doubling bifurcation and intermittency, respectively. In the time interval $0<\tau<\tau_1$, the solutions are chaotic $(\epsilon=0)$; the associated strange attractors are plotted in Figs. 3(b) and 4(b). At $\tau=\tau_1$ a control wave with amplitude $\epsilon=10^{-3}$ is applied and maintained until $\tau\leqslant\tau_2$; in the interval $\tau_1<\tau<\tau_2$ the orbits are periodic with period 1, as seen in the phase portraits in Figs. 3(c) and 4(c). At $\tau=\tau_2$, the control amplitude is changed to $\epsilon=10^{-7}$ and the periodic solutions are modified from period 1 to period 4 for Fig. 3(d), and from period 1 to period 2 for Fig. 4(d).

This method of controlling chaos is fairly simple to carry out in nonlinear three-wave experiments in the laboratory. Once the frequencies and wave vectors of the pump (A_1) and idler (A_2) waves are known, we can apply a small sinusoidal wave having the frequency and wave vector of the anti-Stokes mode $|\epsilon|\sin(\mathbf{k}_4\cdot\mathbf{x}-\omega_4 t)$ to target the chaotic state of the system to a desired periodic orbit by varying the control parameter $|\epsilon|$ appropriately. Our method is based on the fact that the strange attractors, as shown in Figs. 3(b) and 4(b), have embedded in them an infinitive number of unstable periodic orbits [10]. Following the innovative idea first proposed by Ott, Grebogi, and Yorke [9], we have the freedom to choose the most desirable periodic orbit by introducing a small control wave E_4 to the system.

The system (6)–(8) can be targeted to *any* desired periodic orbit through a proper adjustment of the control parameter ϵ. For example, for the chaotic state generated by the period-doubling route [19,20], our method can stabilize *any*

TABLE I. List of periodic orbits stabilized by a small control wave.

Orbit	2^0	2^1	2^2	2^3
ϵ	10^{-3}	10^{-4}	10^{-7}	5×10^{-8}
$\lvert A_1\rvert_{max}$	33.3	37.8	48.5	49.6

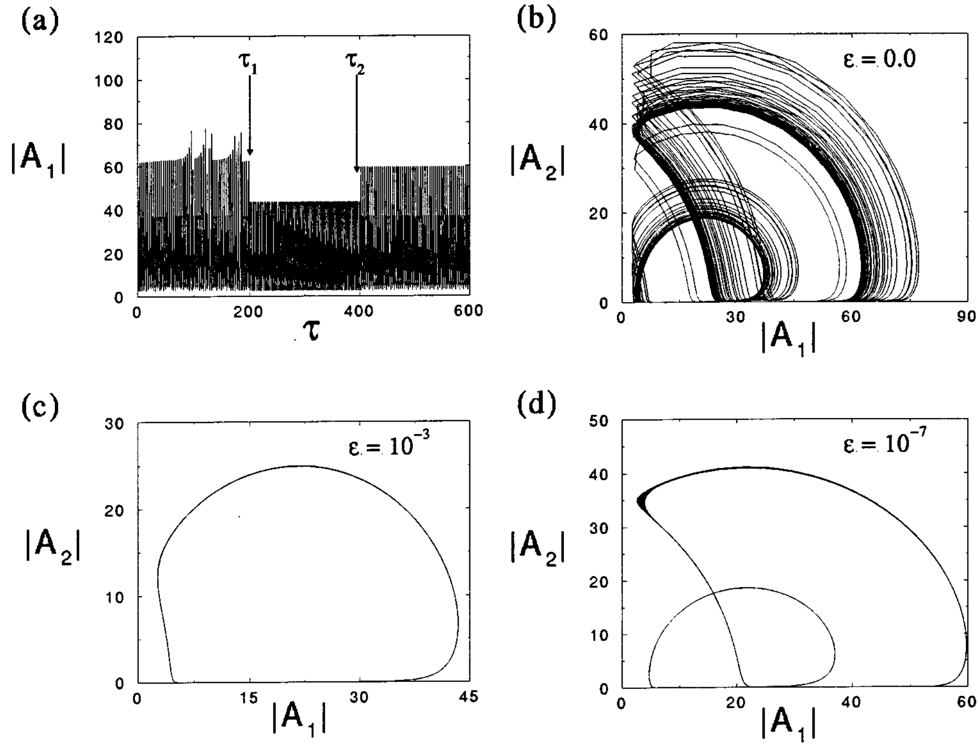

FIG. 4. The plot of (a) the chaotic time series $|A_1(\tau)|$ with $\epsilon=0$ for $0<\tau<\tau_1$, $\epsilon=10^{-3}$ for $\tau_1<\tau<\tau_2$, and $\epsilon=10^{-7}$ for $\tau_2<\tau$; and the corresponding phase portraits for $\epsilon=0$ (b), $\epsilon=10^{-3}$ (c), and $\epsilon=10^{-7}$ (d). The linear frequency mismatch and growth-damping parameters are $\delta=5$ and $\nu=21.61$.

orbit with period 2^n $(n=0,1,2,\dots)$. Table I shows a list of periodic orbits stabilized from the strange attractor of Fig. 3(b), which was obtained as follows. We raise the control parameter until the system stabilizes. Above the threshold for stabilization, as we increase the control parameter the successive orbits with period $2^n, 2^{n-1}, \dots$ are stabilized, as shown in Table I. Note, from Table I, that as n increases the maximum wave amplitude of the pump wave $|A_1|_{\max}$ also increases. This implies that a periodic state with a larger wave amplitude can be achieved by targeting to an orbit with a higher n number. The upper limit of $|A_1|_{\max}$ attainable is determined by the boundary of the attracting basin as seen in Fig. 3(b).

In conclusion, we have shown that the chaotic behaviors of nonlinear three-wave interactions can be controlled via the introduction of a new resonance to the system, by applying a small sinusoidal anti-Stokes wave. This method provides the flexibility of choosing a desired periodic orbit as well as a desired controlled time interval. We believe that this is a simple and effective method of controlling chaos in nonlinear three-wave experiments in plasma physics, nonlinear optics, and fluid mechanics.

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